Fractional High Order Methods For the Nonlinear Fractional Ordinary Differential Equation

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Abstract

In this paper, we consider the nonlinear fractional order ordinary differential equation ${}_0D_t^{\alpha}y(t) = f(y,t)$, $(t > 0)$, $n-1 < \alpha \leq n$, $y^{(i)}(0) = y_0^{(i)}$, $i = 0, 1, 2, \cdots, n-1$, where $f(y,t)$ satisfies the L-condition, i.e., $|f(y_1,t) - f(y_2,t)| \leq L|y_1 - y_2|$ in $t \in [0,T]$. Fractional order linear multiple step methods are introduced. The high order (2-6) approximations of the fractional order ordinary differential equation with initial value are proposed. The consistence, convergence and stability of the fractional high order methods are proved. Finally, some numerical examples are provided to show that the fractional high order methods for solving the fractional order nonlinear ordinary differential equation are computationally efficient solution methods.

Key words: fractional high order methods; nonlinear fractional order; consistence; convergence; stability

1 Introduction

Various fields of science and engineering deal with dynamical systems, which can be described by fractional-order equations. This topic has received a great

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deal of attention in the last decade [7,15,22–24], especially in the fields of viscoelastic materials [1,10,11,27], electrochemical processes [8], dielectric polarization [28], colored noise [29], anomalous diffusion, signal processing [21], control theory [24], advection and dispersion of solutes in natural porous or fractured media [2,3] and chaos [20]. Djrbasjan et al. [6] considered the Cauchy problem with multi-term fractional derivatives, and proved the Cauchy problem has a unique solution. Kilbas et al. [9] gave solution of Volterra integrodifferential equations with generalized Mittag-Leffler function in the kernels. The main reason for the success of the theory in these cases is that these new fractional-order models are more accurate than integer-order models.

Numerical methods associated with integral order ordinary differential equation, for example, Euler method, linear multiple step method, and so on, have seen treated extensively in the literature. On the other hand, theoretical studies of the numerical methods and the error estimate of fractional order differential equation are quite limited, because theoretical analysis of fractional order numerical methods is very difficult [13,14,16–18,26].

Recently linear FDEs based on the Reimann-Liouvile fractional derivative with general variable coefficients are solved by adapting the decomposition method. Not much has been done for the nonlinear FDE. For example, in the modelling of viscoelasticity, the nonlinear FDE can describing the strain and entropy reacting in response to changes in stress and temperature. In the theory of viscoelasticity, it is well known that the constitutive equations governing these phenomena involve differential equations of fractional order. Since the theory of viscoelasticity is essentially a linear theory, these differential equations are also linear, and therefore they may be solved using rather simple methods. We shall see that, in our situation, we have to replace these linear equations by nonlinear ones. The standard solution methods for the linear equations usually fail in the nonlinear case. Thus we have also developed an algorithm for the numerical solution of our equations.

There have been some attempts to solve linear problems, but a complete analysis has not been given so far [25]. Nonlinear equations have received rather less attention in the literature, partly because many of the model equations proposed have been linearized. More recently, applications have included classes of nonlinear fractional differential equations and this motivates us to consider their effective numerical methods for the solution of nonlinear fractional differential equations. Diethelm et al. have done a lot of excellent works on numerical methods for fractional order ordinary differential equations [4,5]. In this paper, we propose fractional high order numerical methods for the nonlinear fractional ordinary differential equation.

This paper is organized as follows: Section 2 presents fractional high order numerical methods. In section 3, we introduce the notations, definitions, and lemmas which we shall use. In sections 4 to 6, we analyze the error theory of the fractional high order numerical methods for the nonlinear fractional differential equation. The consistence, convergence and stability of the fractional high order numerical methods are proven. In section 7, we give some numerical examples based on the use of the high order (2-6) approximations for solving the nonlinear fractional order ordinary differential equation. We show that the high order approximations are computationally efficient methods. Finally, we give a conclusion based on the algorithm for solving the nonlinear fractional order ordinary differential equation.

2 High order fractional linear multiple step methods

In this paper, we consider the following nonlinear fractional order ordinary equation (NFOODE)

$$ {}_0D_t^{\alpha} y(t) = f(y,t),\ (t>0), y^{(i)}(0) = y_0^{(i)},\ i=0,\ 1,\cdots,n-1, \tag{1}$$

where $n-1<\alpha<n$.

In Eq. (1), $f(y,t)$ satisfies the L-condition, i.e., $|f(y_1,t)-f(y_2,t)| \le L|y_1-y_2|$ in $t\in[0,T]$. Eq. (1) uses Riemann-Liouville derivative of order α, which is defined by

$$ {}_0D_t^{\alpha} f(t) = \frac{d^n}{dt^n}\left(\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}\right).$$

Using the relationship between Grünwald-Letnikov and Riemann-Liouville fractional derivatives ([24]), we have

$$ {}_0D_t^{\alpha} f(t) = \lim_{h\to 0} h^{-\alpha}\Delta_h^{\alpha} f(t) = \lim_{h\to 0} h^{-\alpha}\sum_{j=0}^{[t/h]}(-1)^j\binom{\alpha}{j} f(t-jh),$$

where $0\le n-1<\alpha\le n$. *Lubich* has given the p-th ($p=1,2,3,4,5,$ and 6) approximations [19]

$$ {}_0D_t^{\alpha} y(t) \approx {}_0\widetilde{D_t^{\alpha}} y(t) = h^{-\alpha}\sum_{j=0}^{n}\omega_j^{(\alpha)} y(t_{n-j}) + h^{-\alpha}\sum_{j=0}^{s} w_{nj} y(t_j),\ nh=T. \tag{2}$$

So we obtain the approximations of order 1, 2, 3, 4, 5, and 6 in the form of (2), where the coefficients $\omega_k^{(\alpha)}$ are the coefficients of the Taylor series expansions of the corresponding generating functions $W_p^{(\alpha)}(x)$, where p denote the order of method. When p changes from 1 to 6, the corresponding $W_p^{(\alpha)}(x)$ are as follows:

$$ W_1^{(\alpha)}(x) = (1-x)^{\alpha},$$

$$W_2^{(\alpha)}(x) = (\frac{3}{2} - 2x + \frac{1}{2}x^2)^\alpha,$$
$$W_3^{(\alpha)}(x) = (\frac{11}{6} - 3x + \frac{2}{3}x^2 - \frac{1}{3}x^3)^\alpha,$$
$$W_4^{(\alpha)}(x) = (\frac{25}{12} - 4x + 3x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4)^\alpha,$$
$$W_5^{(\alpha)}(x) = (\frac{137}{60} - 5x + 5x^2 - \frac{10}{3}x^3 + \frac{5}{4}x^4 - \frac{1}{5}x^5)^\alpha,$$
$$W_6^{(\alpha)}(x) = (\frac{147}{60} - 6x + \frac{15}{2}x^2 - \frac{20}{3}x^3 + \frac{15}{4}x^4 - \frac{6}{5}x^5 + \frac{1}{6}x^6)^\alpha.$$

And the starting weights $w_{nj}(j = 1, \cdots, s)$ are independent of h. It can be obtained from system of linear equations,

$$\sum_{j=1}^{s} w_{nj} j^q = \frac{\Gamma(q+1)}{\Gamma(-\alpha+q+1)} n^{q-\alpha} - \sum_{j=1}^{s} \omega_{n-j}^{(\alpha)} j^q, (q = 0, \cdots, s-1),$$

where we can take $s = p$ under certain condition ([19]).

Using the relationship between Grünwald-Letnikov and Riemann-Liouville fractional derivatives, and above discrete approximations to fractional derivative term, we obtain p-order (high order, $p \geq 1$) fractional linear multiple step methods (p-HOFLMSM):

$$h^{-\alpha} \sum_{j=0}^{n} \omega_j^{(\alpha)} y_{n-j} + h^{-\alpha} \sum_{j=0}^{s} w_{nj} y_j = f(y_n, t_n),$$
$$y^{(i)}(0) = y_0^{(i)}, i = 0, 1, \cdots, n-1, \tag{3}$$

where $0 < n - 1 < \alpha \leq n$.

3 Preparation

To motivate the following definitions, firstly we consider the case $\alpha = 1$ in (1). If the p-HOFLMSM ($p = 1$) is applied to the first order ordinary differential equation

$$y'(t) = f(t, y), \ y(0) = y_0.$$

It is well known that the resulting numerical solution can be written as the Euler method where the weight ω_j^1, j=1,2,$\cdots$, are the coefficients of

$$W_1^{(1)}(x) = 1 - x.$$

Similar to the consistence and stability of the Euler method, we can define them for p-HOFLMSM.

Definition 3.1 *p-HOFLMSM is consistent if $T_n(t) = y(t_n) - \widetilde{y_n} \to 0$, as $h \to 0$, where $\widetilde{y}$ is a solution of p-HOFLMSM* (3) *and $y(t)$ is a solution of NFOODE* (1).

Definition 3.2 *p-HOFLMSM is stable if any two solutions y_n and y_n' of p-HOFLMSM* (3)*, which use different initial values f and f', satisfy $|y_n - y_n'| \leq C\|f - f'\|_\infty$, where C is a constant.*

Lemma 3.3 *Suppose that $f(t)$ is sufficiently differentiable in $[a, b]$. Then*

$$h^{-\alpha}\sum_{j=0}^{n}\omega_j^{(\alpha)} f(t_{n-j}) + h^{-\alpha}\sum_{j=0}^{s} w_{nj} f(t_{n-j}) - {}_0D_t^{\alpha} f(t) = O(h^p),$$

uniformly in $t \in [a, b]$, and $\omega_j^{(\alpha)} = O(j^{-\alpha-1})$.

Proof. ([19] p.707).

Lemma 3.4 *Let $\omega_k^{(\alpha)}(k = 0, 1, 2, \cdots)$ is the coefficient of the Taylor series expansions of the corresponding "generating" functions, $W_p^{(\alpha)}(x)$, $d_k(k = 0, 1, 2, \cdots)$ is the coefficient of $1/W_p^{(\alpha)}(x)$, and*

$$A = \begin{pmatrix} \omega_0^{(\alpha)} & \omega_1^{(\alpha)} & \cdots & \omega_n^{(\alpha)} \\ 0 & \omega_0^{(\alpha)} & \cdots & \omega_{n-1}^{(\alpha)} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \omega_0^{(\alpha)} \end{pmatrix}, \quad D = \begin{pmatrix} d_0 & d_1 & \cdots & d_n \\ 0 & d_0 & \cdots & d_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & d_0 \end{pmatrix}.$$

We have $A^{-1} = D$ and $d_k = O(k^{\alpha-1})$.

Proof. Let $AD = B$. Easily we have $b_{i,i} = \omega_0^{(\alpha)} d_0, i = 0, \cdots, n$ and $b_{i,j} = 0, i > j$. And if $i < j$, $b_{i,j} = \omega_0^{(\alpha)} d_{j-i} + \cdots + \omega_{j-i}^{(\alpha)} d_0$. Let $f(x) = W_p^{(\alpha)}(x)/W_p^{(\alpha)}(x)$. It implies that

$$f(x) = (\omega_0^{(\alpha)} + \omega_1^{(\alpha)} x + \cdots) \cdot (d_0 + d_1 x + \cdots)$$

$$= \omega_0^{(\alpha)} d_0 + (\omega_1^{(\alpha)} d_0 + \omega_0^{(\alpha)} d_1)x + \cdots + (\omega_k^{(\alpha)} d_0 + \ldots + \omega_0^{(\alpha)} d_k)x^k + \cdots.$$

From $f(x) = 1$, we have $\omega_0^{(\alpha)} d_0 = 1$, and if $i < j$, $b_{i,j} = 0$. Hence, $AD = B = E$. And from [19], we have $d_k = O(k^{\alpha-1})$.

Lemma 3.5 (Gronwall inequality) *Let $a, b \geq 0$, and $\{\eta_i\}$ satisfy*

$$|\eta_n| \leq b + ah \sum_{i=0}^{n-1} |\eta_i|, \; n = k, k+1, \cdots, nh \leq T,$$

then

$$|\eta_n| \leq e^{aT}(b + akhM_0), \; n \geq k, \; nh \leq T,$$

where $M_0 = \max(|\eta_0|, |\eta_1|, \cdots, |\eta_{k-1}|)$.

Proof.([12])

Looking at the questions of existence and uniqueness of the NFOODE, it can be obtained from the following results that are very similar to the corresponding classical theorems in the case of first-order ordinary differential equation. The following theorems have been proven by Diethelm et al. [4].

Theorem 3.6 (existence) *Assume that* $\Omega = [y_0^{(0)} - \beta, y_0^{(0)} + \beta] \times [0, \chi^*]$ *with some* $\chi^* > 0$ *and some* $\beta > 0$, *and let the function* $f : \Omega \to R$ *be continuous. Furthermore, define* $\chi = min\{\chi^*, \alpha\Gamma(q+1)/\|f\|_\infty\}$. *Then, there exists a function* $y : [0, \chi] \to R$ *solving the initial value problem* (1).

Theorem 3.7 (uniqueness) *Assume that* $\Omega = [y_0^{(0)} - \beta, y_0^{(0)} + \beta] \times [0, \chi^*]$ *with some* $\chi^* > 0$ *and some* $\beta > 0$, *and let the function* $f : \Omega \to R$ *be bounded on* Ω *and fulfill a Lipschitz condition with respect to the first variable; i.e.,*

$$| f(y,t) - f(z,t) | \leq L \mid y - z \mid,$$

with some constant $L > 0$ *independent of t,y, and z. Then denoting* χ *as in Theorem* 3.6, *there exists at most one function* $y : [0, \chi] \to R$ *solving the initial value problem* (1).

4 Consistence

Now we start to present the main results about the error analysis of the fractional order linear multiple step methods.

We consider the error between exact solution $y(t_n)$ and numerical solution $\widetilde{y}_n$ given from $y(t_0), \ldots, y(t_{n-1})$.

Theorem 4.1 *p-HOFLMSM* (3) *for solving NFOODE* (1) *is consistent, and*

$$\widetilde{y}_n - y(t_n) = O(h^{p+\alpha}).$$

Proof. Let $n > s$. We have

$$h^{-\alpha}\omega_0^{(\alpha)}\widetilde{y}_n + h^{-\alpha}\sum_{j=1}^{n}\omega_j^{(\alpha)}y(t_{n-j}) + h^{-\alpha}\sum_{j=0}^{s}w_{nj}y(t_j) = f(\widetilde{y}_n, t_n),$$

$$h^{-\alpha}\omega_0^{(\alpha)}\widetilde{y}_n - h^{-\alpha}\omega_0^{(\alpha)}y(t_n) + h^{-\alpha}\sum_{j=0}^{n}\omega_j^{(\alpha)}y(t_{n-j}) + h^{-\alpha}\sum_{j=0}^{s}w_{nj}y(t_j) = f(\widetilde{y}_n, t_n),$$

$$h^{-\alpha}\omega_0^{(\alpha)}(\widetilde{y}_n - y(t_n)) + {}_0D_t^{\alpha}y(t) - C_nh^p = f(y(t_n), t_n) + f(\widetilde{y}_n, t_n) - f(y(t_n), t_n),$$

$$h^{-\alpha}\omega_0^{(\alpha)}(\widetilde{y}_n - y(t_n)) = C_nh^p + (f(\widetilde{y}_n, t_n) - f(y(t_n), t_n)),$$

$$(\widetilde{y}_n - y(t_n)) = C_nh^{p+\alpha} + \frac{h^{\alpha}}{\omega_0^{(\alpha)}}(f(\widetilde{y}_n, t_n) - f(y(t_n), t_n)),$$

$$\begin{aligned}|\widetilde{y}_n - y(t_n)| &\le C_nh^{p+\alpha} + |\frac{h^{\alpha}}{\omega_0^{(\alpha)}}(f(\widetilde{y}_n, t_n) - f(y(t_n), t_n))| \\ &\le C_nh^{p+\alpha} + \frac{Lh^{\alpha}}{\omega_0^{(\alpha)}}|(\widetilde{y}_n - y(t_n))|,\end{aligned}$$

$$(1 - \frac{Lh^{\alpha}}{\omega_0^{(\alpha)}})|(\widetilde{y}_n - y(t_n))| \le C_nh^{p+\alpha}.$$

As h is sufficiently small, $|(\widetilde{y}_n - y(t_n))| \le Ch^{p+\alpha}$.

5 Convergence

Theorem 5.1 *The p-HOFLMSM* (3) *to NFOODE* (1) *is convergent, and* $|y_n - y(t_n)| = O(h^p)$.

Proof. Let $e_n = y(t_n) - y_n$, and we assume $e_0, e_1, \cdots, e_s$ equal to zero.

From

$$h^{-\alpha}\sum_{j=0}^{n}\omega_j^{(\alpha)}y_{n-j} + h^{-\alpha}\sum_{j=0}^{s}w_{nj}y_j = f(y_n, t_n)$$

and

$$h^{-\alpha}\sum_{j=0}^{n}\omega_j^{(\alpha)}y(t_{n-j}) + h^{-\alpha}\sum_{j=0}^{s}w_{nj}y(t_j) = C_nh^p + f(y(t_n), t_n),$$

we get

$$h^{-\alpha}\sum_{j=0}^{n}\omega_j^{(\alpha)}e_{n-j} + h^{-\alpha}\sum_{j=0}^{s}w_{nj}e_j = C_nh^p + f(y(t_n), t_n) - f(y_n, t_n).$$

Let

$$f(y(t_n), t_n) - f(y_n, t_n) = L_n(y(t_n) - y_n) = L_n e_n.$$

Let $|L_n| \leq L$ and $|C_n| \leq C$ for any n. Now we estimate e_n. We can see that

$$(h^{-\alpha}\omega_0^{(\alpha)} - L_n)e_n + h^{-\alpha}\omega_1^{(\alpha)}e_{n-1} + \cdots + h^{-\alpha}\omega_{n-(s+1)}^{(\alpha)}e_{s+1}$$

$$= C_n h^p - h^{-\alpha}\sum_{j=0}^{s}(w_{nj} + \omega_{n-j}^{(\alpha)})e_j,$$

$$(h^{-\alpha}\omega_0^{(\alpha)} - L_{n-1})e_{n-1} + h^{-\alpha}\omega_1^{(\alpha)}e_{n-2} + \cdots + h^{-\alpha}\omega_{n-(s+2)}^{(\alpha)}e_{s+1}$$

$$= C_{n-1}h^p - h^{-\alpha}\sum_{j=0}^{s}(w_{n-1,j} + \omega_{n-1-j}^{(\alpha)})e_j,$$

$$\cdots\cdots$$

$$(h^{-\alpha}\omega_0^{(\alpha)} - L_{s+1})e_{s+1} = C_{s+1}h^p - h^{-\alpha}\sum_{j=0}^{s}(w_{s+1,j} + \omega_{s+1-j}^{(\alpha)})e_j.$$

Let

$$A = \begin{pmatrix} \omega_0^{(\alpha)} & \omega_1^{(\alpha)} & \cdots & \omega_{n-(s+1)}^{(\alpha)} \\ 0 & \omega_0^{(\alpha)} & \cdots & \omega_{n-(s+2)}^{(\alpha)} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \omega_0^{(\alpha)} \end{pmatrix},$$

$$\overrightarrow{e} = \begin{pmatrix} e_n \\ e_{n-1} \\ \cdots \\ e_{s+1} \end{pmatrix}, \overrightarrow{b} = \begin{pmatrix} b_n \\ b_{n-1} \\ \cdots \\ b_{s+1} \end{pmatrix}, B = \begin{pmatrix} L_n & 0 & \cdots & 0 \\ 0 & L_{n-1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & L_{s+1} \end{pmatrix},$$

$$A^{-1} = D = \begin{pmatrix} d_0 & d_1 & \cdots & d_{n-(s+1)} \\ 0 & d_0 & \cdots & d_{n-(s+2)} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & d_0 \end{pmatrix}.$$

Write $h^{-\alpha}A\overrightarrow{e} - B\overrightarrow{e} = \overrightarrow{b}$. We have

$$\overrightarrow{e} - h^{\alpha}DB\overrightarrow{e} = h^{\alpha}D\overrightarrow{b},$$

$$\begin{pmatrix} e_n \\ e_{n-1} \\ \cdots \\ e_{s+1} \end{pmatrix} - \begin{pmatrix} L_n h^{(\alpha)} d_0 & L_{n-1} h^{(\alpha)} d_1 & \cdots & L_{s+1} h^{(\alpha)} d_{n-(s+1)} \\ 0 & L_{n-1} h^{(\alpha)} d_0 & \cdots & L_{s+2} h^{(\alpha)} d_{n-(s+2)} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & L_{n-(s+1)} h^{(\alpha)} d_0 \end{pmatrix} \begin{pmatrix} e_n \\ e_{n-1} \\ \cdots \\ e_{s+1} \end{pmatrix}$$

$$= \begin{pmatrix} h^{(\alpha)} d_0 & h^{(\alpha)} d_1 & \cdots & h^{(\alpha)} d_{n-(s+1)} \\ 0 & h^{(\alpha)} d_0 & \cdots & h^{(\alpha)} d_{n-(s+2)} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & h^{(\alpha)} d_0 \end{pmatrix} \begin{pmatrix} b_n \\ b_{n-1} \\ \cdots \\ b_{s+1} \end{pmatrix}.$$

Consider the first term of $\overrightarrow{e}$ and $\overrightarrow{b}$. We obtain

$$e_n - (L_n h^\alpha d_0, L_{n-1} h^\alpha d_1, \ldots, L_{s+1} h^\alpha d_{n-(s+1)}) \begin{pmatrix} e_n \\ e_{n-1} \\ \cdots \\ e_{s+1} \end{pmatrix}$$

$$= (h^\alpha d_0, h^\alpha d_1, \ldots, h^\alpha d_{n-(s+1)}) \begin{pmatrix} b_n \\ b_{n-1} \\ \cdots \\ b_{s+1} \end{pmatrix},$$

$$(1 - L_n h^\alpha d_0) e_n = \sum_{i=0}^{n-(s+1)} h^\alpha d_i b_{n-i} + \sum_{i=1}^{n-(s+1)} L_{n-i} h^\alpha d_i e_{n-i}.$$

When $\alpha > 1$,

$$|h^{\alpha-1} d_i| \leq C |n^{1-\alpha} i^{\alpha-1}| \leq C,$$

so

$$|\sum_{i=1}^{n} L_{n-1} h^\alpha d_i e_{n-i}| \leq Ch \sum_{i=1}^{n} |L_{n-i} e_{n-i}|.$$

And because $d_i = O(i^{\alpha-1})$,

$$\lim_{n\to\infty} \sum_{i=1}^{n} |h^\alpha d_i| \leq \lim_{n\to\infty} \sum_{i=1}^{n} C' \frac{i^{\alpha-1}}{n^\alpha} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} C' (\frac{i}{n})^{\alpha-1} = C' \int_0^1 x^{\alpha-1} dx = C,$$

then

$$|\sum_{i=1}^{n} h^{\alpha} d_i b_{n-i}| \le C \sup(|b_i|) \le C h^p.$$

So we have

$$\begin{aligned}
|(1 - L_n h^{\alpha} d_0) e_n| &\le C h^p + \sum_{i=1}^{n-(s+1)} |L_{n-i} h^{\alpha} d_i e_{n-i}| \\
&\le C h^p + h \sum_{i=1}^{n-(s+1)} |L_{n-i} h^{\alpha-1} d_i e_{n-i}| \\
&\le C h^p + hLC \sum_{i=1}^{n-(s+1)} |e_{n-i}| \\
&\le C h^p + hLC \sum_{i=s+1}^{n-1} |e_i|.
\end{aligned}$$

As h is sufficiently small,$|e_n| \le Ch^p + hLC \sum_{i=s+1}^{n-1} |e_i|$. Using *Gronwall* inequality, we can prove that $e_n = O(h^p)$.

6 Stability

Theorem 6.1 *Let y_n and y'_n are numerical solutions, which the right hand sides are given by f and f', respectively. Then $|y_n - y'_n| \le C\|f - f'\|_\infty$ for any n, i.e., the p-HOFLMSM* (3) *to the NFOODE* (1) *is stable for the term of the right hand side.*

Proof. Let $y_n - y'_n = \varepsilon_n$.

From

$$h^{-\alpha} \sum_{j=0}^{n} \omega_j^{(\alpha)} y_{n-j} + h^{-\alpha} \sum_{j=0}^{s} w_{nj} y_j = f(y_n, t_n)$$

and

$$h^{-\alpha} \sum_{j=0}^{n} \omega_j^{(\alpha)} y'_{n-j} + h^{-\alpha} \sum_{j=0}^{s} w_{nj} y'_j = f'(y'_n, t_n),$$

we have

$$\begin{aligned}
&h^{-\alpha} \sum_{j=0}^{n} \omega_j^{(\alpha)} \varepsilon_{n-j} + h^{-\alpha} \sum_{j=0}^{s} w_{nj} \varepsilon_{n-j} \\
&= f(y_n, t_n) - f(y'_n, t_n) + f(y'_n, t_n) - f'(y'_n, t_n).
\end{aligned}$$

Let

$$f(y_n, t_n) - f(y'_n, t_n) = L_n(y_n - y'_n) = L_n \varepsilon_n,$$

$$f(y'_n, t_n) - f'(y'_n, t_n) = \Delta f_n,$$

and

$$|f(y,t) - f'(y,t)| \le \Delta f.$$

Obviously $|L_n| \le L$ for any n.

Now we estimate ε_n, we can see that

$$\begin{pmatrix} h^{(-\alpha)}\omega_0^{(\alpha)} - L_n & h^{(-\alpha)}\omega_1^{(\alpha)} & \cdots & h^{(-\alpha)}\omega_{n-(s+1)}^{(\alpha)} \\ 0 & h^{(-\alpha)}\omega_0^{(\alpha)} - L_{n-1} & \cdots & h^{(-\alpha)}\omega_{n-(s+2)}^{(\alpha)} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & h^{(-\alpha)}\omega_0^{(\alpha)} - L_{s+1} \end{pmatrix} \begin{pmatrix} \varepsilon_n \\ \varepsilon_{n-1} \\ \cdots \\ \varepsilon_{s+1} \end{pmatrix}$$
$$= \begin{pmatrix} \Delta f_n \\ \Delta f_{n-1} \\ \cdots \\ \Delta f_{s+1} \end{pmatrix}.$$

Note that $h^{-\alpha}A - B\overrightarrow{\varepsilon} = \overrightarrow{b}$. From the proof above, we easily have

$$(1 - L_n h^\alpha d_0)\varepsilon_n = \sum_{i=0}^{n-(s+1)} h^\alpha d_i b_{n-i} + \sum_{i=1}^{n-(s+1)} L_{n-i} h^\alpha d_i \varepsilon_{n-i}.$$

When $\alpha > 1$,

$$|(1 - L_n h^\alpha d_0)\varepsilon_n| \le |\sum_{i=0}^{n-(s+1)} h^\alpha d_i b_{n-i}| + \sum_{i=1}^{n-(s+1)} |L_{n-i} h^\alpha d_i \varepsilon_{n-i}|,$$

we have

$$|(1 - L_n h^\alpha d_0)\varepsilon_n| \le C\Delta f + \sum_{i=1}^{n-(s+1)} |L_{n-i} h^\alpha d_i \varepsilon_{n-i}| \le C\Delta f + hCL \sum_{i=s+1}^{n-1} |\varepsilon_i|.$$

Hence, $|\varepsilon_n| \le C\Delta f + hCL \sum_{i=s+1}^{n-1} |\varepsilon_i|$.

Using *Gronwall* inequality, we have $|\varepsilon_n| \le C\Delta f + Ch|\varepsilon_{s+1}| \le \bar{C}\Delta f$.

Therefore, for the p-HOFLMSM, if there is a perturbation in right hand side $f(y,t)$, the small change would not cause the large error in the numerical solution. Thus the p-HOFLMSM is stable on right hand term $f(y,t)$.

7 Numerical examples

In this section we consider an initial value problem for one of the fractional order differential equations appearing in applied problems [24]. We would like to present some numerical examples to show the effect of the p-HOFLMSM.

Example 1: Fractional oscillation equation with $\alpha = 1.5$:

$$ {}_0D_t^{\frac{3}{2}} y(t) + y(t) = te^{-t}, y^{(k)}(0) = 0, (k = 0, 1). $$

The exact solution is

$$ y(t) = \int_0^t G(t - x)xe^{-x}dx,\ G(t) = t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}), $$

where $E_{\alpha,\beta}(z)$ is the so-called *Mittag-Leffler* function

$$ E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \alpha > 0,\ \beta > 0. $$

A comparison of the p-HOFLMSM ($p = 3$) and the exact solution is listed in Table 1 and is shown in Figure 1. From Table 1 and Figure 1, it can be seen that our numerical method (3-HOFLMSM) is in excellent agreement with the exact solution and the error between p-HOFLMSM and exact solution is $O(h^p)$.

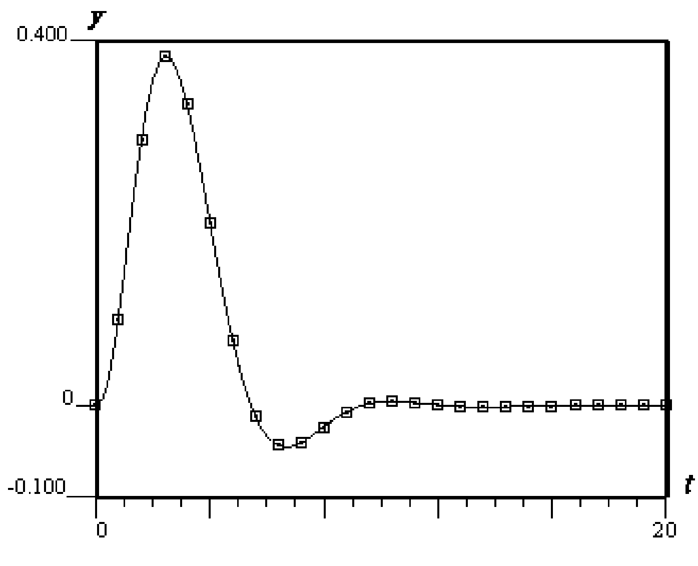

Fig. 1. A comparison of the p-HOFLMSM ($p = 3$) and exact solution in Example 1.

Example 2: Fractional oscillation equation with $1 < \alpha \leq 2$.

Table 1
Numerical solution and error ($n = 100, h = 0.2$)

t	3-HOFLMSM	Exact solution	Error
0.8	9.52875892E-02	9.92045725E-02	3.91698327E-03
2.4	3.84651536E-01	3.84983333E-01	3.31797377E-04
4.0	2.01589390E-01	2.01226046E-01	3.63344850E-04
5.6	-1.20968485E-02	-1.23575173E-02	2.60668883E-04
7.2	-4.09010633E-02	-4.09463462E-02	4.52828845E-05
8.8	-7.20316073E-03	-7.18831888E-03	1.48418475E-05
10.4	4.52776157E-03	4.50733840E-03	2.04231665E-05
12.0	4.73026337E-04	4.27574713E-04	4.54516234E-05
13.6	-1.99206193E-03	-2.03395168E-03	4.18897439E-05
15.2	-1.22253777E-03	-1.25642181E-03	3.38840349E-05
16.8	-3.70895123E-04	-4.03356068E-04	3.24609452E-05
18.4	-2.36528966E-04	-2.70331545E-04	3.38025783E-05
20.0	-2.84607314E-04	-3.18767867E-04	3.41605531E-05
The maximum error is $4.13691139E-03$ in $t = 0.6$			

We are interested in the behaviors of the solution for the fractional oscillation equation:

$$ {}_0D_t^{\alpha} y(t) + y(t) = te^{-t},\ 1 < \alpha \leq 2,\ y^{(i)}(0) = 0,\ i = 0, 1. $$

We show the behaviors of the solutions with $1 < \alpha \leq 1.5$ and $1.5 < \alpha \leq 2$ in Figures 2 and 3, respectively.

Example 3 A nonlinear fractional order ordinary differential equation with $\alpha = 1.5$:

$$ {}_0D_t^{\frac{3}{2}} y(t) + y^2(t) = f(t), y^{(i)}(0) = 0,\ i = 0, 1 $$

where

$$ f(t) = \frac{\Gamma(6)}{\Gamma(6-\alpha)} t^{5-\alpha} - \frac{3\Gamma(5)}{\Gamma(5-\alpha)} t^{4-\alpha} + \frac{2\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha} + (t^5 - 3t^4 + 2t^3)^2. $$

The exact solution is $y(t) = t^5 - 3t^4 + 2t^3$.

We apply the p-HOFLMSM with step sizes $h = 0.02$ and $p = 3$ to obtain the approximation. A comparison of the p-HOFLMSM ($p = 3$) and the exact solution is listed in Table 2 and is shown in Figure 4. From Table 2 and Figure 4, it can be seen that our numerical method (3-HOFLMSM) is in excellent

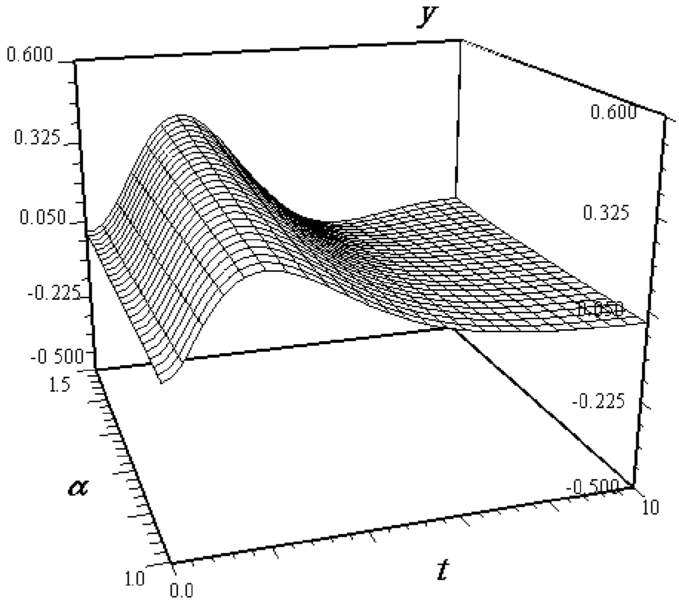

Fig. 2. Behaviors of solution with $1 \leq \alpha \leq 1.5$.

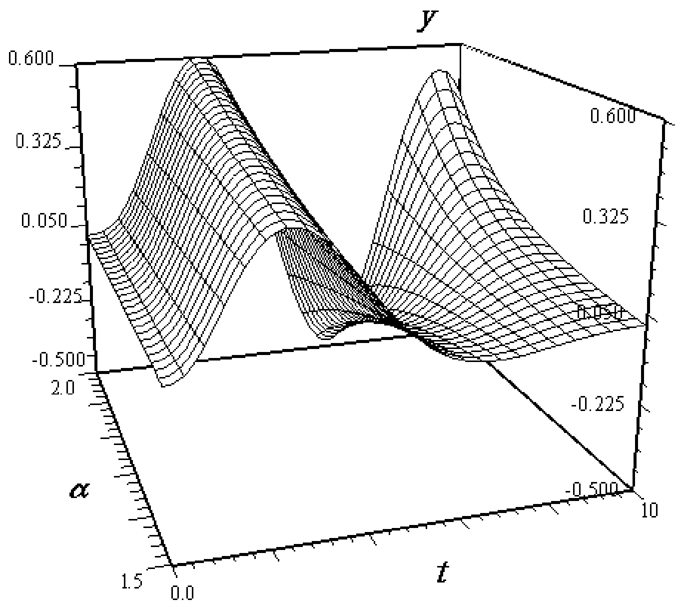

Fig. 3. Behaviors of solution with $1.5 \leq \alpha \leq 2.0$.

agreement with the exact solution and the error between p-HOFLMSM and exact solution is $O(h^p)$.

Example 4 A nonlinear fractional order ordinary differential equation with $1 < \alpha \leq 2$:

$$ {}_0D_t^{\alpha} y(t) + y^2(t) = f(t), $$

where $f(t) = e^{-t}\sin(t), t \in [0, 6]$.

Figure 5 show the behaviors of the nonlinear solution with $1 < \alpha \leq 2$.

Table 2
Numerical solution and error ($n = 100, h = 0.02$)

t	Numerical solution	Exact solution	Error
0.2	1.15167781E-02	1.15200000E-02	-3.22187462E-06
0.4	6.14108077E-02	6.14399999E-02	-2.91922377E-05
0.6	0.12092028	0.12096000	-3.97172672E-05
0.8	0.12284510	0.12288000	-3.48980138E-05
1.0	-1.54449952E-05	0.00000000	-1.54449952E-05
1.2	-0.27646186	-0.27648000	1.81399204E-05
1.4	-0.65849166	-0.65856000	6.83334800E-05
1.6	-0.98289629	-0.98304000	1.43708888E-04
1.8	-0.93286240	-0.93311999	2.57594614E-04
2.0	4.05999983E-04	0.00000000	4.05999983E-04
The maximum error is $4.05999983E-04$ in $t = 2.0$			

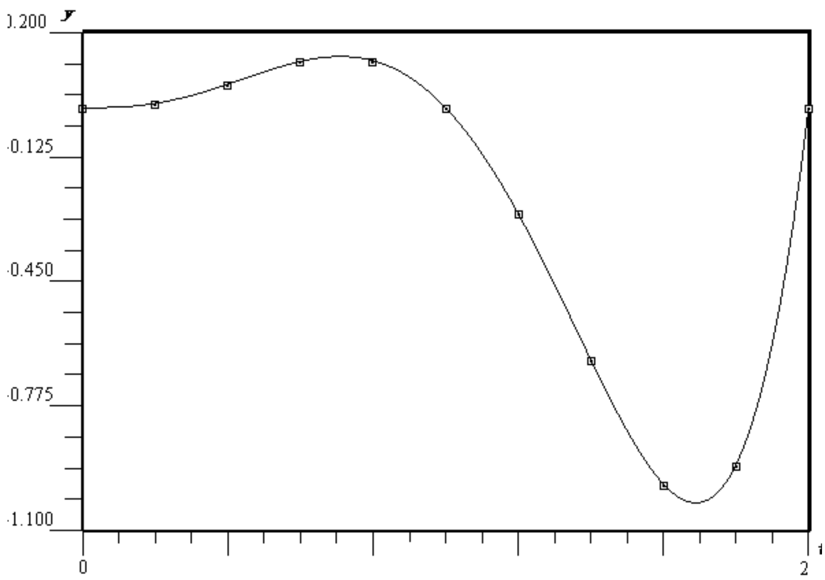

Fig. 4. A comparison of the p-HOFLMSM ($p = 3$) and exact solution in Example 3

8 Conclusion

Fractional high order methods for a fractional order nonlinear ordinary differential equation have been described. The consistence, convergence and stability are proved. Numerical examples are presented. p-HOFLMSM for solving the fractional order ordinary differential equation is an effective method.

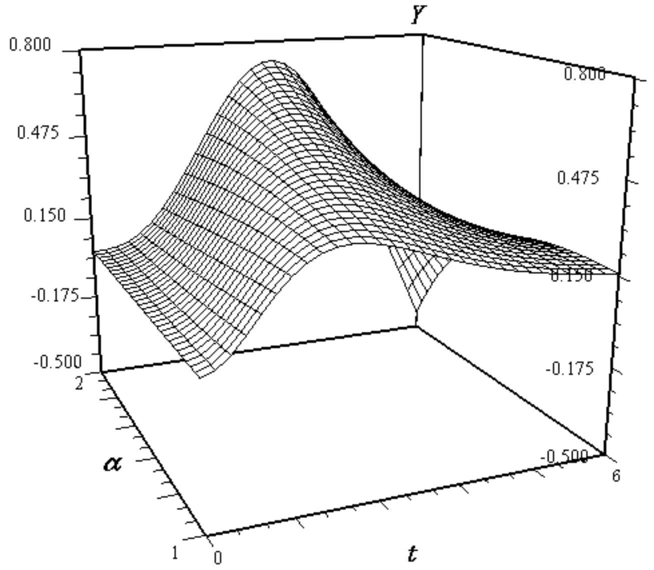

Fig. 5. behaviors of the nonlinear solution with $1 < \alpha \leq 2$

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